Properties of convergence in Dirichlet structures

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Abstract

In univariate settings, we prove a strong reinforcement of the energy image density criterion for local Dirichlet forms admitting square field operators. This criterion enables us to redemonstrate classical results of Dirichlet forms theory [1]. Besides, when $X = (X_1, \ldots, X_p)$ belongs to the $\mathbb{D}$ domain of the Dirichlet form, and when its square field operator matrix $\Gamma[X, {}^tX]$ is almost surely definite, we prove that $\mathcal{L}_X$ is Rajchman. This is the first result in full generality in the direction of Bouleau-Hirsch conjecture. Moreover, in multivariate settings, we study the particular case of Sobolev spaces: we show that a convergence for the Sobolev norm $\mathcal{W}^{1,p}(\mathbb{R}^d, \mathbb{R}^p)$ toward a non-degenerate limit entails convergence of push-forward measures in the total variation topology. [7].

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1. Introduction

In the two past decades, the theory of Dirichlet forms has been extensively studied in the direction of improving regularity results of Malliavin calculus (cf.[14],[13],[8]). Usually, Malliavin calculus enables integrating by parts in order to prove, for any test functions, inequalities of the form $\left|\mathbb{E}(\phi^{(p)}(X))\right| \leq C\|\phi\|_{C^0}$, from which smoothness estimates on the law of X may be derived. Nevertheless, when one only wants to prove the absolute continuity with respect to Lebesgue measure (without quantitative estimates), a more efficient tool exists: the energy image density criterion. More precisely, in the particular setting of local Dirichlet forms admitting square field operators, the energy image density E.I.D. asserts that a random variable $X = (X_1, \cdots, X_p)$ in the $\mathbb{D}^p$ domain of the Dirichlet form, which square field operator matrix $\Gamma[X] = (\Gamma[X_i, X_j])_{1\leq i,j\leq p}$ is almost surely definite ($\det \Gamma[X] > 0$), possesses a density with respect to the Lebesgue measure on $\mathbb{R}^n$. Concretely, with respect to Malliavin calculus approach, this criterion

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enables to weaken the assumptions of smoothness, but all the more to relax the non degeneracy conditions from $\frac{1}{\det\Gamma[X]} \in L^Q(\Omega)$ $(Q > 1)$ to $\det\Gamma[X] \neq 0$. Always true for real valued variables $(p = 1)$, this criterion is still a conjecture in the general context since 1986 (cf. [7]). Nevertheless, this criterion has been established in almost all practically encountered Dirichlet forms such as the Ornstein-Uhlenbeck form on the Wiener space (cf. [7]), or Dirichlet forms on the Poisson space (cf. [10]). Besides, very recently, a new method called "the lent particle" uses E.I.D. to provide with a powerful tool enabling to study the regularity of the laws of Levy processes or solutions of stochastic differential equations driven by Levy processes (cf. [4],[5],[6]). In the above cases, the energy image density criterion is established, using some rather involved arguments of geometric measure theory around the coarea formula for mappings belonging to Sobolev spaces $W^{1,2}(\mathbb{R}^d)$. Since theses arguments use the particular topology of $\mathbb{R}^d$ (local compactness), they cannot be transposed to the general case of a local Dirichlet form $\mathcal{E}$ acting on a sub-domain of an abstract $L^2(\mathbb{P})$ (instead of $L^2(k(x)d\lambda_d(x))$ space, which concerns most of infinite dimensional contexts. Until now, the univariate case is the only result enabling to prove tha absolute continuity in full generality (cf. [5], chap.5).

In this article, we set a new property, namely the strong energy image density S.E.I.D., which is a kind of quantitative version of E.I.D.. More precisely, in univariate cases and for general Dirichlet forms, we prove that if X_n converge toward X in the $\mathbb{D}$ domain of a Dirichlet structure, then $X_{n*}\left(\mathbb{1}_{\{\Gamma[X]>0\}}d\mathbb{P}\right)$ converges in total variation toward $X_*\left(\mathbb{1}_{\{\Gamma[X]>0\}}d\mathbb{P}\right)$. As a corollary, we redemonstrate the fact that convergence for the Dirichlet topology is preserved by Lipschitz mappings in the univariate case, which has been established in [1]. Besides, in the multivariate case, we prove that the distribution of $X \in \mathbb{D}^p$ whose square field operator matrix is almost surely definite, is a Rajchman measure. To our knowledge, this is the first result in full generality in the direction of Bouleau-Hirsch conjecture. Finally, we study multivariate cases for the $\mathcal{W}^{1,p}(\Omega)$ Sobolev structures (Ω open subset or $\mathbb{R}^d$) and establish a weaker form of S.E.I.D. but with rather precise estimates. Indeed, we need some additional assumptions in order to ensure that the Jacobian of the mappings is integrable, which enables us to prove an integration by parts formula. Let us mention that our proof (Lemma 4.3) provides with a generalization of a recent result from H.Brezis and H.M.Nguyen (c.f. [9] Theorem1). Besides, theses results are extended to the case of the Ornstein-Uhlenbeck form in the Wiener space giving a generalization (by a purely algebraic method and avoiding every coarea type arguments) of classical results from Bouleau-Hirsch [8]. Every results of this article are heavily inspired by the seminal paper [7], where functional calculus and completeness of $\mathbb{D}$ are combined, in order to prove E.I.D.. As explained in 2.4, our proof has a common structure with [7]. Roughly speaking, in [7], the preponderant argument is the completeness of $\mathbb{D}$ whereas in ours proofs we use the infinitesimal generator A (inducing the Dirichlet form $\mathcal{E}[\cdot,\cdot]$). Although theses two facts are totally equivalent (c.f.

[11]), we highlight that using the generator is a more powerful approach of E.I.D.. We are grateful to Vlad Bally and Damien Lamberton for their advices notably in concern with the proof of 4.1.

After having precised the notations and stated the main results, we focus on the univariate case for a general structure. Next, we end by studying the particular case of Sobolev spaces $\mathcal{W}^{1,p}(\Omega)$.

2. Notations and results

2.1. Definitions and notations

Originally introduced by Beurling and Deny (c.f. [2]), a Dirichlet form is a symmetric non-negative bilinear form $\mathcal{E}[\cdot,\cdot]$ acting on a dense subdomain $\mathcal{D}(\mathcal{E})$ of an Hilbert space $\mathcal{H}$, such that $\mathcal{D}(\mathcal{E})$ endowed with the norm $\sqrt{< X, X >_{\mathcal{H}} + \mathcal{E}[X, X]}$ is complete. We refer to [12, 11, 8, 3] for an exhaustive introduction to this theory. In the sequel we only focus on the particular case of local Dirichlet forms admitting square field operators. In order to avoid unessential difficulties, we restrict our attention to the case of probability spaces $(\Omega, \mathcal{F}, \mathbb{P})$ instead of measured spaces $(\Omega, \mathcal{F}, m)$. The next definition is central.

Definition 2.1. *Following the terminology of [8], in this paper, a Dirichlet structure will denote a term* $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{D}, \Gamma)$ *such that:*

(a) $(\Omega, \mathcal{F}, \mathbb{P})$ *is a probability space.*

(b) $\mathbb{D}$ *is a dense subdomain of* $L^2(\mathbb{P})$.

(c) $\Gamma[\cdot,\cdot] : \mathbb{D} \times \mathbb{D} \longrightarrow L^1(\mathbb{P})$ *is bilinear, symmetric, non-negative.*

(d) For all $m \geq 1$, *for all* $X = (X_1, \cdots, X_m) \in \mathbb{D}^m$, *and for all* $F \in \mathcal{C}^1(\mathbb{R}^m, \mathbb{R})$ *and* K*-Lipschitz:*

- $F(X) \in \mathbb{D}$,
- $\Gamma[F(X), F(X)] = \sum_{i=1}^{m}\sum_{j=1}^{m} \partial_i F(X)\partial_j F(X)\Gamma[X_i, X_j].$

(e) Setting $\mathcal{E}[X, X] = \mathbb{E}\left(\Gamma[X, X]\right)$, *the domain* $\mathbb{D}$ *endowed with the norm:*

$$\|X\|_{\mathbb{D}} = \sqrt{\mathbb{E}(X^2) + \mathcal{E}[X, X]},$$

is complete. Thus, $\mathcal{E}$ *is a Dirichlet form with domain* $\mathbb{D}$ *on the Hilbert space* $L^2(\mathbb{P})$.

Let us recall briefly, that there exists an operator A defined on a $\mathcal{D}(A)$ dense subdomain of $L^2(\mathbb{P})$ such that:

1. $U \in \mathcal{D}(A)$ if and only if there exists $C_U > 0$ satisfying:
$$\forall X \in \mathbb{D},\ |\mathbb{E}\{\Gamma[X,U]\}| \leq C_U \sqrt{\mathbb{E}\{X^2\}},$$
2. for all $U \in \mathcal{D}(A)$, for all $X \in \mathbb{D}$ and all $Z \in \mathbb{D} \cap L^\infty(\mathbb{P})$, we have:
$$\mathbb{E}\{\Gamma[X,U]Z\} = -\mathbb{E}\{XZA[U]\} + \mathbb{E}\{X\Gamma[Z,U]\},$$
3. $\mathcal{D}(A)$ is dense in $\mathbb{D}$ for the norm $\|\cdot\|_{\mathbb{D}}$.

Let us enumerate the notations adopted in the present paper:

- for $X \in \mathbb{D}$, we set $\Gamma[X] = \Gamma[X,X]$ and $\mathcal{E}[X] = \mathcal{E}[X,X]$,
- for $X = (X_1, \cdots, X_m) \in \mathbb{D}^m$:
$$\Gamma[X] = \Gamma[X, {}^tX] = \begin{pmatrix} \Gamma[X_1,X_1] & \Gamma[X_1,X_2] & \cdots & \Gamma[X_1,X_n] \\ \Gamma[X_2,X_1] & \Gamma[X_2,X_2] & \cdots & \Gamma[X_2,X_n] \\ \vdots & \vdots & \cdots & \vdots \\ \Gamma[X_n,X_1] & \Gamma[X_n,X_2] & \cdots & \Gamma[X_n,X_n] \end{pmatrix},$$
- for $\phi \in \mathcal{C}^1(\mathbb{R}^d, \mathbb{R})$, we set $\vec{\nabla}\phi(x) = (\partial_1\phi(x), \ldots, \partial_d\phi(x))$,
- in a topological space $(E, \mathcal{T})$, $x_n \xrightarrow[n\to\infty]{\mathcal{T}} x$ naturally means that x_n converges toward x in the topology $\mathcal{T}$,
- for a random variable X taking values in $\mathbb{R}^p$, $\mathcal{L}_X$ is the ditribution of X and $\hat{\mathcal{L}}_X(\xi)$ its characteristic function,
- for a Radon measure μ, we set $\|\mu\|_{TV} = \sup_{\{\|\phi\|_{\mathcal{C}^0} \leq 1\}} < \phi, \mu >$ the total variation of μ,
- finally, in the spaces $\mathbb{R}^p$, $\|\cdot\|$ will be the Euclidean norm.

The following definition is preponderant in this work.

Definition 2.2. *Let $S = (\Omega, \mathcal{F}, \mathbb{P}, \mathbb{D}, \Gamma)$ be a Dirichlet structure. We say that S satisfies the energy image density criterion, if and only if, for all $p \geq 1$, for all $X = (X_1, \cdots, X_p) \in \mathbb{D}^p$:*
$$X_*(\mathbb{1}_{\{\det \Gamma[X] > 0\}} d\mathbb{P}) << d\lambda_p.$$

Conjecture. *(Bouleau-Hirsch)*
Every Dirichlet structure (in the sense of 2.1) satisfies the criterion E.I.D..

As already mentioned, we refer to [7, 10, 5] for examples and sufficient conditions entailing E.I.D.. The most illustrative example of this kind of structure is the Sobolev space $\mathcal{H}^1(\Omega, \lambda_d)$ where Ω is a bounded open subset of $\mathbb{R}^d$, and λ_d the d-dimensional Lebesgue measure In this case:

- $(\Omega, \mathcal{F}, \mathbb{P}) = (\Omega, \mathcal{B}(\Omega), \frac{d\lambda_d}{\lambda_d(\Omega)})$,
- $\mathbb{D} = \mathcal{H}^1(\Omega)$,
- $\Gamma[\phi] = \vec{\nabla}\phi \,.\, {}^t\vec{\nabla}\phi$,
- $\mathcal{E}[\phi] = \dfrac{1}{\lambda_d(\Omega)} \displaystyle\int_\Omega \vec{\nabla}\phi {}^t\vec{\nabla}\phi \, d\lambda_d$.

2.2. Results in general Dirichlet structures

Theorem 2.1. *Let $S = (\Omega, \mathcal{F}, \mathbb{P}, \mathbb{D}, \Gamma)$ be a Dirichlet structure. Let $(X_n)_{n\in\mathbb{N}}$ and X be in the domain $\mathbb{D}$. We assume that $X_n \xrightarrow[n\to\infty]{\mathbb{D}} X$, then for any $Z \in L^1(\mathbb{P})$ supported in $\{\Gamma[X] > 0\}$:*

$$\sup_{\|\phi\|_{\mathcal{C}^0}\leq 1} \mathbb{E}\left\{(\phi(X_n) - \phi(X))\, Z\right\} \xrightarrow[n\to\infty]{} 0. \tag{1}$$

For instance, if almost surely $\Gamma[X] > 0$, then $\|\mathcal{L}_{X_n} - \mathcal{L}_X\|_{TV} \xrightarrow[n\to\infty]{} 0$.

Corollary 2.1. *Let $S = (\Omega, \mathcal{F}, \mathbb{P}, \mathbb{D}, \Gamma)$ be a Dirichlet structure and let $(X_n)_{n\in\mathbb{N}}$ and X be in $\mathbb{D}$. If $X_n \xrightarrow[n\to\infty]{\mathbb{D}} X$ then for any Lipschitz map F, $F(X_n) \xrightarrow[n\to\infty]{\mathbb{D}} F(X)$.*

Theorem 2.2. *Let $S = (\Omega, \mathcal{F}, \mathbb{P}, \mathbb{D}, \Gamma)$ a Dirichlet structure and let $X_n = (X_n^{(1)}, \cdots, X_n^{(p)})$ and $X = (X^{(1)}, \cdots, X^{(p)})$ be in $\mathbb{D}^p$. Besides, we assume that almost surely $\det\Gamma[X] > 0$ and that $X_n \xrightarrow[n\to\infty]{\mathbb{D}^p} X$. Then:*

$$\sup_{\xi\in\mathbb{R}^p} \left|\hat{\mathcal{L}}_{X_n}(\xi) - \hat{\mathcal{L}}_X(\xi)\right| \xrightarrow[\|\xi\|\to\infty]{} 0. \tag{2}$$

Particularly, we get that $\hat{\mathcal{L}}_X(\xi) \xrightarrow[\|\xi\|\to\infty]{} 0$, that is to say $\mathcal{L}_X$ is a Rajchman measure.

2.3. Results in Sobolev structures

Theorem 2.3. *Let $d \geq p$ be two integers, and Ω be an open subset of $\mathbb{R}^d$. (F_n) be a sequence in $\mathcal{W}^{1,p}(\Omega, \mathbb{R}^p)$ converging to F in $\mathcal{W}^{1,p}(\Omega, \mathbb{R}^p)$ and K in $L^1(\Omega, \mathbb{R})$ a measurable map supported in $\{\det\Gamma[F] \neq 0\}$, where $\Gamma[F] = \vec{\nabla}F \cdot {}^t\vec{\nabla}F$. Then*

$$(F_n)_*(Kd\lambda) \xrightarrow[n\to+\infty]{VT} F_*(Kd\lambda)$$

Corollary 2.2. *We endow $\mathbb{R}^{\mathbb{N}}$ with the Gaussian probability $\mathbb{P} = \mathcal{N}(0,1)^{\mathbb{N}}$ Let p an integer, (F_n) a sequence in $\mathcal{W}^{1,p}(\mathbb{P})$ converging to F in $\mathcal{W}^{1,p}(\mathbb{P})$. Let I a finite subset of $\mathbb{N}$ with cardinal p and K in $L^1(\mathbb{P})$ supported in $\{\det \Gamma(F) \neq 0\}$ where $\Gamma[F] = \vec{\nabla} F \cdot {}^t\vec{\nabla} F$. Then*

$$(F_n)_*(KdP) \xrightarrow[n\to+\infty]{T.V.} F_*(KdP)$$

Remark 2.1. *Previous corollary ensures that if $X_n = (X_1^{(n)}, \cdots, X_p^{(n)})$ converges to $X = (X_1, \cdots, X_p)$ in $\mathbb{D}^{1,p}$ and if $\det \Gamma[X] > 0$ then $\mathcal{L}_{X_n}$ converges toward $\mathcal{L}_X$ in the total variation topology. Here $\mathbb{D}^{1,p}$ is the analogue of the space $\mathcal{W}^{1,p}$ for the Malliavin gradient operator D, and Γ is the square field operator associated to the Ornstein-Uhlenbeck form in the Wiener space (c.f. [14]). This can be applied easily to the case of diffusions on $\mathbb{R}^p$ with $\mathcal{C}^1$ coefficients, where dependence theorems on the initial conditions will ensure that $X_t^x \xrightarrow[x\to y]{\mathbb{D}^{1,p}} X_t^y$. For instance, applying Proposition 3.1, we redemonstrate the classical result of Bouleau-Hirsch in [7].*

In the same way as Corollary 2.1, we can deduce from Theorem 2.3 that the convergence in $\mathcal{W}^{1,p}(\Omega)$ is preserved by Lipschitz mappings. But in this case contrarly to Corollary 2.1, a non-degeneresence of the limit is required.

Corollary 2.3. *Let $d \geq p$ be two integers, Ω be an open set of $\mathbb{R}^d$ and (F_n), F belong to $\mathcal{W}^{1,p}(\Omega, \mathbb{R}^p)$ such that*

$$\begin{cases} F_n \xrightarrow[n\to+\infty]{\mathcal{W}^{1,p}} F \\ \det \Gamma[F] = \det\left(\vec{\nabla} F \cdot {}^t\vec{\nabla} F\right) > 0 \text{ a.e.} \end{cases}.$$

Then for any Lipschitz mapping $\Phi : \mathbb{R}^p \mapsto \mathbb{R}$:

$$\Phi \circ F_n \xrightarrow[n\to+\infty]{\mathcal{W}^{1,p}} \Phi \circ F.$$

2.4. Scheme of the proofs

Our proofs of Theorem 2.1 and 2.3 have a similar structure, constituted of three steps. Roughly speaking:

- We give us a bounded continuous function φ and we set $\Phi(x) = \int^x \varphi(t)dt$ a primitive. (If $\phi \in \mathcal{C}(\mathbb{R}^d, \mathbb{R})$ with d>1) we only integrate with respect to the one variable).
- Using an integration by parts formula (involving Φ and functional calculus (c.f. (d) 2.1), we obtain for W "smooth" an inequality on the form :

$$|\mathbb{E}[\varphi(X_n)J_nW] - \mathbb{E}[\varphi(X)JW]| \leq C(W)\|\varphi\|_{C^0}\|X_n - X\|_{L^Q}$$

  where J_n (resp. J) are playing the role of $\Gamma[X_n]$ (resp. $\Gamma[X]$) in the univariate case and the role of $\det \Gamma[X]$ (resp. $\det \Gamma[X_n]$) in the multivariate setting.

- Assumptions will ensure that when n tend to $+\infty$, J_n will be close to J, so that choosing $W \approx \frac{1}{J}$, we will obtain the smallness of the quantity $\mathbb{E}[\varphi(X_n)] - \mathbb{E}[\varphi(X)]$, uniformly in $\|\varphi\|_0$.

Let us mention that in the multivariate setting, our integration by parts relies on Schwartz property $(\partial_x \partial_y = \partial_y \partial_x)$, and we failed in finding an analogue in the general context.

3. Strong energy image density in the general setting

Lemma 3.1. *For all $M > 0$, let $E_M = \{\phi \in \mathcal{C}_c^0([-M, M]) \mid \|\phi\|_{\mathcal{C}^0} \leq 1\}$. Then for any sequence ϕ_n in E_M, for any $U \in \mathcal{D}(A)$ and any $W \in L^\infty(\mathbb{P}) \cap \mathbb{D}$:*

$$\lim_{n \to \infty} \mathbb{E}\left\{(\phi_n(X_n) - \phi_n(X))\, \Gamma[X, U] W\right\} = 0.$$

Proof. Let us be placed under the assumptions of the lemma. Using functional calculus we have:

$$\Gamma[\Phi_n(X_n) - \Phi_n(X), U] = \phi_n(X_n)\Gamma[X_n, U] - \phi_n(X)\Gamma[X, U]. \tag{3}$$

Cauchy-Schwarz inequality entails:

$$\mathbb{E}\left\{|\Gamma[X_n, U] - \Gamma[X, U]|\right\} \leq \sqrt{\mathcal{E}[X_n - X]}\sqrt{\mathcal{E}[U]}. \tag{4}$$

Functional calculus (3) and inequality (4) ensure that:

$$\begin{aligned} &|\mathbb{E}\left\{\Gamma[\Phi_n(X_n) - \Phi_n(X), U] W\right\} - \mathbb{E}\left\{(\phi_n(X_n) - \phi_n(X))\, \Gamma[X, U] W\right\}| \\ &\leq \|W\|_\infty \sqrt{\mathcal{E}[X_n - X]}\sqrt{\mathcal{E}[U]}. \end{aligned} \tag{5}$$

Moreover, the usual integration by parts leads to:

$$\mathbb{E}\left\{\Gamma[\Phi_n(X_n) - \Phi_n(X), U] W\right\} = \mathbb{E}\left\{(\Phi_n(X_n) - \Phi_n(X))\left(\Gamma[U, W] - A[U] W)\right)\right\}. \tag{6}$$

Finally using both (5) and (6), we get:

$$\begin{aligned} |\mathbb{E}\left\{(\phi_n(X_n) - \phi_n(X))\, \Gamma[X, U] W\right\}| &\leq \mathbb{E}\left\{|\Phi_n(X_n) - \Phi_n(X)|\, |\Gamma[U, W] - A[U] W)|\right\} \\ &\leq \mathbb{E}\left\{|X_n - X| \wedge (2M)\, |\Gamma[U, W] - A[U] W)|\right\} \\ &\quad + \|W\|_\infty \sqrt{\mathcal{E}[X_n - X]}\sqrt{\mathcal{E}[U]}. \end{aligned} \tag{7}$$

Since, $|\Gamma[U, W] - A[U] W)| \in L^1(\mathbb{P})$, inequality (7) ensures that:

$$\lim_{n \to \infty} \mathbb{E}\left\{(\phi_n(X_n) - \phi_n(X))\, \Gamma[X, U] W\right\} = 0.$$

□

Proof. *(Theorem 2.1)*

Let M be a positive constant and let us fix a sequence ϕ_n in E_M such that:

$$\left| \sup_{\phi\in E_M} \mathbb{E}\left\{(\phi(X_n)-\phi(X))\,Z\right\} - \mathbb{E}\left\{(\phi_n(X_n)-\phi_n(X))\,Z\right\} \right| \leq \frac{1}{n}.$$

Up to extracting a subsequence, we assume that $(\phi_n(X_n)-\phi_n(X))$ converges weakly in $L^\infty(\mathbb{P})$ toward $Y\in L^\infty(\mathbb{P})$. In particular , for all $U\in\mathcal{D}(A)$ and all $W\in\mathcal{D}\cap L^\infty(\mathbb{P})$:

$$\mathbb{E}\left\{(\phi_n(X_n)-\phi_n(X))\,\Gamma[X,U]W\right\} \xrightarrow[n\to\infty]{} \mathbb{E}\left\{Y\Gamma[X,U]W\right\}.$$

Using Lemma 3.1, we deduce $\mathbb{E}\left\{Y\Gamma[X,U]W\right\}=0$. Besides, $\mathcal{D}(A)$ is dense in $\mathbb{D}$ for the norm $\|\cdot\|_{\mathbb{D}}$ and $\mathbb{D}\cap L^\infty(\mathbb{P})$ is dense in $L^1(\mathbb{P})$ so that $Y\Gamma[X]=0$ a.s. and hence $YZ=0$ a.s., and in consequence,

$$\lim_{n\to\infty}\sup_{\phi\in E_M}\mathbb{E}\left\{(\phi(X_n)-\phi(X))\,Z\right\} = \mathbb{E}\left\{YZ\right\}=0.$$

In the general case, let us notice that:

$$\begin{aligned}\Bigg|\sup_{\phi\in E_M}\mathbb{E}\left\{(\phi(X_n)-\phi(X))\,\mathbb{1}_{\{\Gamma[X]>0\}}Z\right\} - \sup_{\|\phi\|_{\mathcal{C}^0}\leq 1}\mathbb{E}\left\{(\phi(X_n)-\phi(X))\,\mathbb{1}_{\{\Gamma[X]>0\}}Z\right\}\Bigg| \\ \leq \mathbb{E}\left\{\left(\mathbb{1}_{\{|X_n|>M\}}+\mathbb{1}_{\{|X|>M\}}\right)Y\right\} \xrightarrow[M\to\infty]{} 0.\end{aligned}$$

□

Remark 3.1. *Let us notice that the conclusion of Theorem 2.1 remains if we replace the assumption* $X_n \xrightarrow[n\to\infty]{\mathbb{D}} X$ *by the weaker assumption*

$$\begin{cases} X_n \xrightarrow[n\to\infty]{\textit{Prob.}} X \\ \forall U\in\mathcal{D}(A),\ \Gamma[X_n,U] \xrightarrow[n\to\infty]{L^1(\mathbb{P})} \Gamma[X,U] \end{cases}.$$

Now let us prove Corollary 2.1:

Proof. *(Corollary 2.1)*

Let us be placed under the assumptions of Corollary 2.1, let f be a Borel representation of F' and let K be the Lipschitz constant of F. First, as the mapping F is K-Lipschitz, it is straightforward that:

$$F(X_n) \xrightarrow[n\to+\infty]{L^2(\mathbb{P})} F(X).$$

For real valued variables of $\mathbb{D}$, the E.I.D. criterion enables Lipschitz functional calculus. Thus, we have:

$$\begin{aligned}\Gamma[F(X_n)-F(X)] &= f(X_n)^2\Gamma[X_n]+f(X)^2\Gamma[X]-2f(X_n)f(X)\Gamma[X_n,X],\\ &= (f(X)-f(X_n))^2\Gamma[X]+R_n,\end{aligned}$$

with

$$\mathbb{E}\left\{|R_n|\right\}\leq K^2\mathbb{E}\left\{|\Gamma[X_n]-\Gamma[X]|\right\}+2K^2\sqrt{\mathcal{E}[X_n-X]}\sqrt{\mathcal{E}[X]}\xrightarrow[n\to\infty]{}0.$$

Moreover,

$$\begin{aligned}\mathbb{E}\left\{(f(X)-f(X_n))^2\,\Gamma[X]\right\} &= \mathbb{E}\left\{\left(f(X_n)^2-f(X)^2\right)\Gamma[X]\right\}+2\mathbb{E}\left\{f(X)\Gamma[X]\left(f(X)-f(X_n)\right)\right\}\\ &= A_n+B_n.\end{aligned}$$

By Theorem 2.1, we know that for every bounded Borelian mapping $\phi:\mathbb{R}\to\mathbb{R}$ and every Z in $L^1(\mathbb{P})$ supported in $\{\Gamma[X]\neq 0\}$,

$$\mathbb{E}\left\{\left(\phi(X_n)-\phi(X)\right)Z\right\}\xrightarrow[n\to\infty]{}0$$

In particular:

- For $Z=\Gamma[X]$ and $\phi=f^2$, we obtain that $\lim\limits_{n\to\infty}A_n=0$.
- For $Z=f(X)\Gamma[X]$ and $\phi=f$, we obtain that $\lim\limits_{n\to\infty}B_n=0$.

In conclusion:

$$\lim_{n\to\infty}\mathbb{E}\left\{\Gamma[F(X_n)-F(X)]\right\}=0$$

and the corollary is proven. □

Now we come to the proof of Theorem (2.2), it is the first result in full generality in the direction of proving the conjecture of Bouleau-Hirsch. In order to exploit the uniformity provided by Theorem (2.1), we will use linear combinations of the variables $(X_1,\cdots,X_p)$ and $(X_n^{(1)},\cdots,X_n^{(p)})$.

Proof. Theorem 2.2

Let us be placed under the assumptions of the Theorem (2.2), and let ξ_n be in $\mathbb{R}^p$ such that:

$$\left|\sup_{\xi\in\mathbb{R}^p}\mathbb{E}\left\{e^{i<X_n,\xi>}-e^{i<X,\xi>}\right\}-\mathbb{E}\left\{e^{i<X_n,\xi_n>}-e^{i<X,\xi_n>}\right\}\right|\leq\frac{1}{n}. \qquad (8)$$

Now, we rewrite the right term:

$$\mathbb{E}\left\{e^{i<X_n,\xi_n>} - e^{i<X,\xi_n>}\right\} = \mathbb{E}\left\{e^{i\|\xi_n\|<X_n,\frac{\xi_n}{\|\xi_n\|}>} - e^{i\|\xi_n\|<X,\frac{\xi_n}{\|\xi_n\|}>}\right\}. \qquad (9)$$

By compactness of the Euclidean p-sphere, up to extracting a subsequence, we may assume that $\frac{\xi_n}{\|\xi_n\|} \xrightarrow[n\to\infty]{} \xi$, where $\|\xi\| = 1$. We then deduce that:

$$< X_n, \frac{\xi_n}{\|\xi_n\|} > \xrightarrow[n\to\infty]{\mathbb{D}} < X, \xi > .$$

Besides, $\Gamma[< X, \xi >] = \xi\Gamma[X]^t\xi > 0$. We are under the assumptions of Theorem 2.1 by choosing $\phi_n(x) = e^{i\|\xi_n\|x}$, and thanks to (9) we get $\lim\limits_{n\to\infty} \mathbb{E}\left\{e^{i<X_n,\xi_n>} - e^{i<X,\xi_n>}\right\} = 0$. Using inequality (8):

$$\sup_{\xi\in\mathbb{R}^p} \left|\hat{\mathcal{L}}_{X_n}(\xi) - \hat{\mathcal{L}}_X(\xi)\right| \xrightarrow[n\to\infty]{} 0.$$

In order to prove that $\mathcal{L}_X$ is Rajchman, it is enough to make a convolution. More precisely, let $Y_n = X + \frac{1}{n}(U_1, \cdots, U_p)$ where U_i are i.i.d. with common law $\mathbb{1}_{[0,1]}(x)dx$. In fact, the variables U_i may be thought as the coordinates of the usual Dirichlet structure:

$$([0, 1], \mathcal{B}([0, 1]), \mathbb{1}_{[0,1]}(x)dx, \mathcal{H}^1([0, 1]), \Gamma[\phi] = (\phi')^2)^p.$$

In addition, $Y_n \xrightarrow[n\to\infty]{\widetilde{\mathbb{D}}} X$ where $\widetilde{\mathbb{D}}$ is the domain of the product structure:

$$S \times ([0, 1], \mathcal{B}([0, 1]), \mathbb{1}_{[0,1]}(x)dx, \mathcal{H}^1([0, 1]), \Gamma[\phi] = (\phi')^2)^p.$$

Being absolutely continuous, $\hat{\mathcal{L}}_{Y_n}(\xi) \xrightarrow[\|\xi\|\to\infty]{} 0$. We may use the first part of the Theorem which ensures that:

$$\sup_{\xi\in\mathbb{R}^p} \left|\hat{\mathcal{L}}_{Y_n}(\xi) - \hat{\mathcal{L}}_X(\xi)\right| \xrightarrow[n\to\infty]{} 0$$

to complete the proof. □

Remark 3.2. *The aforementioned Rajchman property is consequence on some" uniformity" on the linear combinations of the coordinates of X_n and X. Without this uniformity and using only E.I.D. we may only prove that:*

$$\forall\xi \in \mathbb{R}^p/\{0\}, \lim_{t\to\infty} \mathbb{E}\left\{e^{it<\xi,X>}\right\} = 0.$$

We guess that we could use a wider class of functions than linear ones, in order to operate on the coordinates of X_n and X, the operating class of functions needing to be finite dimensional so that the same compactness argument holds. For instance we could have chosen the class of polynomial maps with degree less than N, in order to get a stronger property than the Rajchman one. Unfortunately, we failed in using this argument.

Theorem 2.1 suggests to introduce the following definition which is a generalization to the multivariate case.

Definition 3.1. *Let $S = (\Omega, \mathcal{F}, \mathbb{P}, \mathbb{D}, \Gamma)$ a Dirichlet structure. We will say that S satisfies the "strong" energy image density E.I.D., if for any $X = (X_1, \cdots, X_p)$ and any sequence $X_n = (X_n^{(1)}, \cdots, X_n^{(p)})$ in $\mathbb{D}^p$ with $\|X_n - X\|_{\mathbb{D}^p} \xrightarrow[n\to\infty]{} 0$ and for any $Z \in L^1(\mathbb{P})$:*

$$\sup_{\|\phi\|_{\mathcal{C}^0(\mathbb{R}^p,\mathbb{R})}\leq 1} \mathbb{E}\left\{(\phi(X_n) - \phi(X))\, \mathbb{1}_{\{\det\Gamma[X]>0\}} Z\right\} \xrightarrow[n\to\infty]{} 0. \tag{10}$$

The terminology of E.I.D. is justified by the next proposition.

Proposition 3.1. *Let $S = (\Omega, \mathcal{F}, \mathbb{P}, \mathbb{D}, \Gamma)$ be a Dirichlet structure satisfying S.E.I.D., then it satisfies the criterion E.I.D..*

Proof. Let A be a Borel subset of $\mathbb{R}^n$ negligible with respect to the Lebesgue measure and let $X = (X_1, \cdots, X_n)$ be in $\mathbb{D}^n$. Let us be given $\hat{U} = (\hat{U}_1, \cdots, \hat{U}_n)$ n random variables i.i.d. with common law $\mathbb{1}_{[0,1]}(x)dx$ and defined on an independent probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}})$. Let us take $Z = 1$ and let us apply E.I.D.:

$$\lim_{n\to\infty} \hat{\mathbb{E}}\left\{\mathbb{E}\left\{\left(\chi_A(X + \frac{1}{n}\hat{U}) - \chi_A(X)\right)\mathbb{1}_{\{\det\Gamma[X]>0\}}\right\}\right\} = 0. \tag{11}$$

But Fubini theorem entails:

$$\hat{\mathbb{E}}\left\{\mathbb{E}\left\{\chi_A(X + \frac{1}{n}\hat{U})\right\}\right\} = \mathbb{E}\left\{\hat{\mathbb{E}}\left\{\chi_A(X + \frac{1}{n}\hat{U})\right\}\right\} = 0.$$

Finally, (11) ensures that $\mathbb{E}\left\{\chi_A(X)\mathbb{1}_{\{\det\Gamma[X]>0\}}\right\} = 0$. □

In the next section, we prove a weaker version of E.I.D. criterion in the Sobolev spaces $W_1^p(\mathbb{R}^p)$. Our result is weaker because we need stronger moments on the variables $\Gamma[X]$. Fortunately, in every practically encountered cases, this assumption is fulfilled. Besides, in the particular setting of standard Sobolev spaces, we are able to prove several generalisations of the S.E.I.D. criterion.

4. Strong energy image density in the Sobolev spaces

In this section, we will prove Theorem 2.3. Let us be placed under the assumptions of the theorem. If $I = \{i_1, \dots, i_p\}$ is a subset of $\{1, \dots, d\}$, for f in $\mathcal{W}^{1,p}(\Omega, \mathbb{R})$ we denote $\vec{\nabla}_I f = (\partial_{i_1} f, \dots, \partial_{i_p} f)$, and for $F = (f_1, \dots, f_p)$ in $\mathcal{W}^{1,p}(\Omega, \mathbb{R}^p)$, we denote $\mathcal{J}_I(F) = \det(\vec{\nabla}_I f_1, \dots, \vec{\nabla}_I f_d)$. With these notations, we have

$$\det\Gamma[F] = \sum_{\substack{I\subset\{1,\dots,d\}\\ |I|=p}} |\mathcal{J}_I(F)|^2.$$

Thus, up to replacing K by $K\mathbb{1}_{\{\mathcal{J}_I(F)\neq 0\}}$, we will assume that K is supported in $\{\mathcal{J}_I(F) \neq 0\}$ from some subset $I = \{i_1, \ldots, i_p\}$.

The integration by part formula used in this section is the following:

Lemma 4.1. *For $f_1, \ldots, f_p$ in $\mathcal{W}^{1,p}(\Omega, \mathbb{R})$ and w in $C^1_c(\Omega, \mathbb{R})$:*

$$\int_\Omega w \det(\vec{\nabla}_I f_1, \vec{\nabla}_I f_2, \ldots, \vec{\nabla}_I f_p) d\lambda = -\int_\Omega f_1 \det(\vec{\nabla}_I w, \vec{\nabla}_I f_2, \ldots, \vec{\nabla}_I f_p) d\lambda.$$

Proof. Without loss of generality, we can assume that $I = \{1, \ldots, p\}$. Besides, since $C^2(\Omega, \mathbb{R})$ is dense in $\mathcal{W}^{1,p}(\Omega, \mathbb{R})$ and $(f_1, \ldots, f_p) \mapsto \det(\vec{\nabla}_I f_1, \ldots, \vec{\nabla}_I f_p)$ is continuous from $\mathcal{W}^{1,p}(\Omega, \mathbb{R}^p)$ into $L^1(\Omega, \mathbb{R})$, we can also assume that $f_1, \ldots, f_p$ are C^2 on Ω. Noticing that

$$w \det(\vec{\nabla}_I f_1, \vec{\nabla}_I f_2, \ldots, \vec{\nabla}_I f_p) + f_1 \det(\vec{\nabla}_I w, \vec{\nabla}_I f_2, \ldots, \vec{\nabla}_I f_p) = \det(\vec{\nabla}_I (w f_1), \vec{\nabla}_I f_2, \ldots, \vec{\nabla}_I f_p),$$

and that

$$\begin{aligned} \int_\Omega \det(\vec{\nabla}_I (w f_1), \vec{\nabla}_I f_2, \ldots, \vec{\nabla}_I f_p) d\lambda &= \int_\Omega \left(\sum_{\sigma \in S_p} \varepsilon(\sigma) \partial_{\sigma(1)} f_1 \cdots \partial_{\sigma(p)} f_p \right) d\lambda \\ &= -\int_\Omega w f_1 \left(\sum_{\sigma \in S_p} \varepsilon(\sigma) \partial_{\sigma(1)} (\partial_{\sigma(2)} f_2 \cdots \partial_{\sigma(p)} f_p) \right) d\lambda, \end{aligned}$$

it is sufficient to prove the algebraic relation

$$\sum_{\sigma \in S_p} \varepsilon(\sigma) \partial_{\sigma(1)} (\partial_{\sigma(2)} f_2 \cdots \partial_{\sigma(p)} f_p) = 0.$$

The left term equals $\sum_{k=1}^p h_k$ where

$$h_k = \sum_{\sigma \in S_p} \varepsilon(\sigma) \partial_{\sigma(1)} \partial_{\sigma(k)} f_1 \prod_{j \neq k} \partial_{\sigma(j)} f_j.$$

Denoting by τ the transposition $(1, k)$ we have

$$\begin{aligned} h_k &= \sum_{\sigma \in S_p} \varepsilon(\sigma \circ \tau) \partial_{\sigma \circ \tau(1)} \partial_{\sigma \circ \tau(k)} f_1 \prod_{j \neq k} \partial_{\sigma(j)} f_j \\ &= -\sum_{\sigma \in S_p} \varepsilon(\sigma) \partial_{\sigma(k)} \partial_{\sigma(1)} f_1 \prod_{j \neq k} \partial_{\sigma(j)} f_j \\ &= -h_k \end{aligned}$$

so that h_k is null, which completes the proof. $\square$

Using this integration by parts, we will establish this new one more general, relating integral of $\varphi \circ F$ and $\partial_1 \varphi \circ F$ against some weights.

Lemma 4.2. *For $F = (f_1, \ldots, f_p)$ in $\mathcal{W}^{1,p}(\Omega, \mathbb{R}^p)$, φ in $C^1(\mathbb{R}^p, \mathbb{R})$ and w in $C^1_c(\Omega, \mathbb{R})$:*

$$\int_\Omega (\partial_1 \varphi \circ F) w \mathcal{J}_I(F) d\lambda = - \int_\Omega \varphi \circ F \det(\vec{\nabla}_I w, \vec{\nabla}_I f_2, \ldots, \vec{\nabla}_I f_p) d\lambda.$$

Proof. Let $\Phi = (\varphi, q_2, \ldots, q_p)$, where q_k is the k-th canonical projection on $\mathbb{R}^d$. Then $\Phi \circ F = (\varphi \circ F, f_2, \ldots, f_p)$, and Lemma 4.1 applied to $\varphi \circ F, f_2, \ldots, f_p$ leads to

$$\int_\Omega w \mathcal{J}_I(\Phi \circ F) d\lambda = - \int_\Omega \varphi \circ f \det(\vec{\nabla}_I w, \vec{\nabla}_I f_2, \ldots, \vec{\nabla} f_p) d\lambda.$$

Noticing that $\mathcal{J}_I(\Phi) = \partial_1 \varphi$ we have

$$\mathcal{J}_I(\Phi \circ F) = (\mathcal{J}_I(\Phi) \circ F) \mathcal{J}_I(F) = (\partial_1 \varphi \circ F) \mathcal{J}_I(F),$$

which completes the proof.

□

Lemma 4.3. *For $F = (f_1, \ldots, f_p)$ and $G = (g_1, \ldots, g_p)$ in $\mathcal{W}^{1,p}(\Omega, \mathbb{R}^p)$, for ψ in $C^0(\mathbb{R}^p, \mathbb{R})$ with $\|\psi\|_{C^0} \leq 1$, for w in $C^1_c(\Omega, \mathbb{R})$ and for r,s positive numbers with $\frac{1}{r} + \frac{p-1}{s} = 1$:*

$$\left| \int_\Omega (\psi \circ F) \mathcal{J}_I(F) w d\lambda - \int_\Omega (\psi \circ G) \mathcal{J}_I(G) w d\lambda \right| \leq C \|F - G\|_{L^r(\Omega)}$$

where

$$C = \sup_{1 \leq k \leq p} (\|\vec{\nabla}_I f_k\|_{L^s(\Omega)} + \|\vec{\nabla}_I g_k\|_{L^s(\Omega)})^{p-1} \|\vec{\nabla}_I w\|_\infty.$$

Remark 4.1. *For $\psi = 1$, it gives Brezis estimate [9]*

Proof. Let us denote $H_k = (f_1, \ldots, f_k, g_{k+1}, \ldots g_p)$ for $k = 0, \ldots, p$. Since

$$(\psi \circ F) \mathcal{J}_I(F) - (\psi \circ G) \mathcal{J}_I(G) = \sum_{k=0}^{p-1} (\psi \circ H_{k+1}) \mathcal{J}_I(H_{k+1}) - (\psi \circ H_k) \mathcal{J}_I(H_k),$$

up to replacing (F, G) by (H_k, H_{k+1}), we may assume that F and G differ from only one coordinate, namely j. Besides, up to a permutation, we may assume $j = 1$, so that $f_k = g_k$ for $k \neq 1$. We set

$$\varphi(x_1, \ldots, x_p) = \int_0^{x_1} \psi(t, x_2, \ldots, x_p) dt.$$

Then by Lemma 4.2,

$$\left\{ \begin{array}{l} \displaystyle\int_\Omega (\psi \circ F) w \mathcal{J}_I(F) d\lambda = \int_\Omega \varphi \circ F \det(\vec{\nabla}_I w, \vec{\nabla}_I f_2, \dots, \vec{\nabla}_I f_p) d\lambda \\ \displaystyle\int_\Omega (\psi \circ G) w \mathcal{J}_I(G) d\lambda = \int_\Omega \varphi \circ G \det(\vec{\nabla}_I w, \vec{\nabla}_I f_2, \dots, \vec{\nabla}_I f_p) d\lambda \end{array} \right. ,$$

and hence

$$\left| \int_\Omega (\psi \circ F) \mathcal{J}_I(F) w d\lambda - \int_\Omega (\psi \circ G) \mathcal{J}_I(G) w d\lambda \right| \leq \\ \int_\Omega |\varphi \circ F - \varphi \circ G| \, |\det(\vec{\nabla}_I w, \vec{\nabla}_I f_2, \dots, \vec{\nabla}_I f_p)| d\lambda.$$

Since $|\partial_1 \varphi| = |\psi| \leq 1$, we have

$$|\varphi \circ F(x) - \varphi \circ G(x)| = \left| \int_{f_1(x)}^{g_1(x)} \psi(t, f_2(x), \dots, f_p(x)) dt \right| \leq |f_1(x) - g_1(x)|$$

and Lemma 4.3 is then as a consequence of the Hölder inequality. □

Proof. (Theorem 2.3)

Let $\mu_n = (F_n)_*(K d\lambda)$, $\mu = F_*(K d\lambda)$. We fix w in $C_c^1(\Omega, \mathbb{R})$ supported on some bounded open set Ω', and we write:

$$\int_{\mathbb{R}^p} \varphi d\mu - \int_{\mathbb{R}^p} \varphi d\mu_n = I_1 + I_2 + I_3$$

with

$$\left\{ \begin{array}{l} \displaystyle I_1 = \int_{\Omega'} (\varphi \circ F) w \mathcal{J}_I(F) d\lambda - \int_{\Omega'} (\varphi \circ F_n) w \mathcal{J}_I(F_n) d\lambda \\ \displaystyle I_2 = \int_{\Omega'} (\varphi \circ F_n) w (\mathcal{J}_I(F_n) - \mathcal{J}_I(F)) d\lambda \\ \displaystyle I_3 = \int_\Omega (\varphi \circ F - \varphi \circ F_n)(K - w \mathcal{J}_I(F)) d\lambda. \end{array} \right.$$

Since (F_n) is bounded in $W^{1,p}(\Omega')$, using Lemma 4.3 with $r = s = p$ we can bound I_1 by $C \|F_n - F\|_{L^p(\Omega')}$ where C does not depend on n. Besides, integrals I_2 and I_3 are respectively less than $2\|\mathcal{J}_I(F_n) - \mathcal{J}_I(F)\|_{L^1(\Omega')}$ and $\|w J_I(F) - K\|_{L^1(\Omega)}$. This leads to

$$\|\mu_n - \mu\|_{TV} \leq C \|F_n - F\|_{L^p(\Omega')} + 2\|\mathcal{J}_I(F_n) - \mathcal{J}_I(F)\|_{L^1(\Omega')} + \|w \mathcal{J}_I(F) - K\|_{L^1(\Omega)}$$

Letting n tend to $+\infty$, we get

$$\limsup_{n \to +\infty} \|\mu_n - \mu\|_{TV} \leq \|w J_I(F) - K\|_{L^1(\Omega)}.$$

Since $\frac{K}{\mathcal{J}_I(F)} \in L^1(\Omega, |\mathcal{J}_I(F)|d\lambda)$ and since $C_c^1(\Omega,\mathbb{R})$ is dense in $L^1(\Omega, |\mathcal{J}_I(F)|d\lambda)$, we may choose w such that $\|w\mathcal{J}_I(F) - K\|_{L^1(\Omega)}$ is as small as wished. We deduce:

$$\limsup_{n\to+\infty} \|\mu_n - \mu\|_{TV} = 0.$$

□

Remark 4.2. *The conclusion of Theorem 2.3 would remain true replacing the assumption $F_n \xrightarrow[n\to+\infty]{\mathcal{W}^{1,p}(\Omega)} F$ by*

$$\begin{cases} F_n \xrightarrow[n\to+\infty]{\textit{Prob.}} F \\ \sup_n \|\vec{\nabla}_I F_n\|_{L^{p-1}(\Omega)} < +\infty \\ J_I(F_n) \xrightarrow[n\to+\infty]{L^1(\Omega)} J_I(F) \end{cases}.$$

This is a consequence of the following estimate: for ψ in $C^0(\mathbb{R}^p,\mathbb{R})$ supported in $[-M,M]^p$ such that $\|\psi\|_{C^0} \leq 1$,

$$\left| \int_\Omega (\psi\circ F_n)\mathcal{J}_I(F_n) w d\lambda - \int_\Omega (\psi\circ F)\mathcal{J}_I(F) w d\lambda \right| \leq$$
$$\int_\Omega (\|F_n(x) - F(x)\| \wedge 2M) |\det(\vec{\nabla}_I w, \vec{\nabla}_I f_2, \ldots, \vec{\nabla}_I f_p)| d\lambda.$$

(The proof is similar of Lemma 4.3)
The right term tends to 0 under above assumptions, which next allows to conclude as in Theorem 2.3 proof.

Proof. Corollary 2.2
We fix an integer d larger than $\max I$. For y in $\mathbb{R}^{\mathbb{N}}$ and x in $\mathbb{R}^d$ we denote $i_y(x) = (x,y)$. Up to extracting a subsequence of (F_n), we can assume that for almost every y in $\mathbb{R}^{\mathbb{N}}$,

$$F_n \circ i_y \xrightarrow[n\to+\infty]{W^{1,p}(\mathbb{R}^d, \mathcal{N}(0,1)^d)} F \circ i_y.$$

Thus, for almost every y in $\mathbb{R}^{\mathbb{N}}$, and every ψ in $C^0(\mathbb{R}^{\mathbb{N}},\mathbb{R})$ satisfying $\|\psi\|_{C^0} \leq 1$,

$$\int_{\mathbb{R}^d} \left(\psi\circ(F_n\circ i_y)(x) - \psi\circ(F\circ i_y)(x)\right)(K\circ i_y(x)) e^{-|x|^2} dx =$$
$$\int_{|x|\leq M} \left(\psi\circ(F_n\circ i_y)(x) - \psi\circ(F\circ i_y)(x)\right)(K\circ i_y(x)) e^{-|x|^2} dx$$
$$+ \int_{|x|>M} \left(\psi\circ(F_n\circ i_y)(x) - \psi\circ(F\circ i_y)(x)\right)(K\circ i_y(x)) e^{-|x|^2} dx.$$
$$= A_n + B_n.$$

Using a Markov inequality, for all $\epsilon > 0$ there exist $M > 0$ such that $\sup_{n,\psi} B_n \leq \epsilon$. Besides, A_n tends to 0 as n tends to $+\infty$ uniformly in ψ by Theorem 2.3. Integrating on y, we deduce that:

$$\int_{\mathbb{R}^{\mathbb{N}}} (\psi \circ F_n - \psi \circ F) K d\mathbb{P} = \int_{\mathbb{R}^{\mathbb{N}}} \int_{\mathbb{R}^d} (\psi \circ (F_n \circ i_y)(x) - \psi \circ (F \circ i_y)(x)) (K \circ i_y(x)) e^{-|x|^2} dx d\mathbb{P}(y)$$

tends to 0 uniformly in ψ as n tends to $+\infty$. $\square$

Proof. (Corollary 2.3) Since Φ is Lipchitz it is straightforward that

$$\Phi \circ F_n \xrightarrow[n \to +\infty]{L^p} \Phi \circ F.$$

Next, fixing k in $\{1, \ldots, d\}$, we have

$$\begin{aligned} \partial_k (\Phi \circ F_n - \Phi \circ F) &= \sum_{i=1}^{p} \partial_k f_i^{(n)} (\partial_i \Phi \circ F_n) - \partial_k f_i (\partial_i \Phi \circ F) \\ &= \sum_{i=1}^{p} \partial_k f_i (\partial_i \Phi \circ F_n - \partial_i \Phi \circ F) + R_n \end{aligned}$$

where $R_n \xrightarrow[n \to +\infty]{L^p(\Omega)} 0$.

Now we fix i in $\{1, \ldots, d\}$, and we want to prove that $\partial_k f_i (\partial_i \Phi \circ F_n - \partial_i \Phi \circ F)$ converges to 0 in $L^p(\Omega)$ Since this sequence is clearly bounded in $L^p(\Omega)$, it is sufficient to prove the convergence in $L^p_{\text{loc}}(\Omega)$, and hence we can assume that $\lambda(\Omega) < +\infty$. We will follow the same strategy that in proof of corollary 2.1. First, we write that

$$\|\partial_i \Phi \circ F_n - \partial_i \Phi \circ F\|^2_{L^2(\Omega)} = \int_\Omega (\partial_i \Phi \circ F_n)^2 d\lambda - 2 \int_\Omega (\partial_i \Phi \circ F_n)(\partial_i \Phi \circ F) d\lambda + \int_\Omega (\partial_i \Phi \circ F)^2 d\lambda.$$

By Theorem 2.3, we know that for any bounded borelian map $\varphi : \mathbb{R}^p \mapsto \mathbb{R}$ and any mapping K in $L^1(\Omega)$,

$$\lim_{n \to +\infty} \int_\Omega (\varphi \circ F_n) K d\lambda = \int_\Omega (\varphi \circ F_n) K d\lambda.$$

In particular:

- For $K = 1$ and $\varphi = \partial_i \Phi^2$ we obtain

$$\lim_{n \to +\infty} \int_\Omega (\partial_i \Phi \circ F_n)^2 d\lambda = \int_\Omega (\partial_i \Phi \circ F)^2 d\lambda.$$

- For $K = \partial_i \Phi \circ F$ and $\varphi = \partial_i \Phi$, we obtain

$$\lim_{n \to +\infty} \int_\Omega (\partial_i \Phi \circ F_n)(\partial_i \Phi \circ F) d\lambda = \int_\Omega (\partial_i \Phi \circ F)^2 d\lambda.$$

Thus, we deduce that $\partial_i \Phi \circ F_n$ converges toward $\partial_i \Phi \circ F$ in $L^2(\Omega)$, and hence in $L^p(\Omega)$ by Holder inequality since $\partial_i \Phi$ is bounded.

Finally, writing that

$$\|\partial_k f_i (\varphi \circ F_n - \varphi \circ F)\|_{L^p(\Omega)} \leq 2 \|\partial_k f_i \mathbb{1}_{\{|\partial_k f_i| > M\}}\|_{L^p(\Omega)} + M \|\varphi \circ F_n - \varphi \circ F\|_{L^p(\Omega)}$$

where M is arbitrarily large, we deduce that $\partial_k f_i(\varphi \circ F_n)$ converges toward $\partial_k f_i(\varphi \circ F)$ in $L^p(\Omega)$. $\square$

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